MATH2050C Selected Solution to Assignment 6

Section 3.4

(4a). The subsequence $b_n = a_{2n} = 1/(2n) \to 0$ as $n \to \infty$. On the other hand, the subsequence $c_n = a_{2n+1} = 2 + 1/(2n+1) \to 2$ as $n \to \infty$. Since these two subsequences converge to different limits, $\{a_n\}$ is divergent.

(b). The subsequence $b_k = a_{8k} = \sin 8k\pi/4 = 0$ while the subsequence $c_k = a_{8k+2} = \sin(8k + 2)\pi/4 = 1$. Thus the first subsequence tends to 0 and the second one to 0. We conclude that this sequence is divergent.

(7a). Observe $a_n = (1 + 1/n^2)^{n^2}$ is a subsequence of $c_n = (1 + 1/n)^n$. In fact, $a_n = c_{n^2}$. Since every subsequence converges to the same limit for a convergent sequence, we have $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = e$.

(d). In a previous exercise we have shown that $a_n = (1+2/n)^n$ is convergent (actually when 2 is replaced by any positive a). Denote its limit by a. Then the subsequence $b_k = a_{2k} = (1+1/k)^{2k}$ should tend to the same a. But now it is clear that it converges to e^2, so $a = e^2$. Therefore, $\lim_{n\to\infty} a_n = e^2$.

(11). Let $a_n = (-1)^n x_n$. By assumption it tends to some a. The subsequence $b_k = a_{2k} = x_{2k}$ tends to a, showing that $a \geq 0$. On the other hand, $c_k = a_{2k+1} = -x_{2k+1}$ also tends to a, showing that $a \leq 0$. (Recall it is assumed that all $x_n \geq 0$.) We conclude that $a = 0$. For every $\varepsilon > 0$, there is some n_ε such that $|x_n - 0| = |(-1)^n x_n - 0| < \varepsilon$ for all $n \geq n_\varepsilon$, hence $\{x_n\}$ converges to 0.

Section 3.5

For $m > n$,

$$\begin{aligned} |x_m - x_n| &\leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n| \\ &< r^{m-1} + r^{m-2} + \cdots r^n \\ &= r^n(r^{m-n-1} + r^{m-n-2} + \cdots + 1) \\ &< r^n \sum_{k=0}^{\infty} r^k \\ &= \frac{r^n}{1-r} . \end{aligned}$$

Since $r^n \to 0$, given $\varepsilon > 0$, we can always find some n_ε such that $r^n/(1-r) < \varepsilon$ for all $n \geq n_\varepsilon$. It follows that $|x_m - x_n| < \varepsilon$ for all $m, n \geq n_\varepsilon$. So $\{x_n\}$ is a Cauchy sequence.

Supplementary Exercises

1. Can you find a sequence from $[0, 1]$ with the following property: For each $x \in [0, 1]$, there is subsequence of this sequence taking x as its limit? Suggestion: Consider the rational numbers.

   Solution. Let $\{r_n\}$ be an enumeration of the set of all rational numbers in $[0, 1]$. This is possible as all rational numbers form a countable set. Let $x \in [0, 1]$. We claim that it is a limit point. For each $n \geq 1$, there are infinitely many rational numbers in $(x - 1/n, x + 1/n) \cap [0, 1]$. We can pick one by one from $\{r_n\}$ to form $\{r_{n_k}\}$ so that $n_k < n_{k+1}$, that is, $\{r_{n_k}\}$ is a subsequence. Now, given $\varepsilon > 0$, pick some n_1 such that $1/n_1 < \varepsilon$. It then follows that for all $n_k \geq n_1$, $|r_{n_k} - x| < 1/n_k \leq 1/n_1 < \varepsilon$. We conclude $r_{n_k} \to x$.

Note. This exercise shows that the set of limit points of a single sequence could be very large.

2. The concept of a sequence extends naturally to points in $\mathbb{R}^N$. Taking $N = 2$ as a typical case, a sequence of ordered pairs, $\{\mathbf{a}_n\}, \mathbf{a}_n = (x_n, y_n)$, is said to be convergent to $\mathbf{a}$ if, for each $\varepsilon > 0$, there is some n_0 such that

$$|\mathbf{a}_n - \mathbf{a}| < \varepsilon \ , \quad \forall n \geq n_0 \ .$$

Here $|\mathbf{a}| = \sqrt{x^2 + y^2}$ for $\mathbf{a} = (x, y)$. Show that $\lim_{n\to\infty} \mathbf{a}_n = \mathbf{a}$ if and only if $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$.

Solution. It follows from the elementary inequalities

$$|x_1 - y_1|, |x_2 - y_2| \leq |\mathbf{a} - \mathbf{b}| \leq |x_1 - y_1| + |x_2 - y_2| \ ,$$

which show that $\mathbf{a}_n \to \mathbf{a}$ if and only if $x_n \to x$ and $y_n \to y$.

3. Bolzano-Weierstrass Theorem in $\mathbb{R}^N$ reads as, every bounded sequence in $\mathbb{R}^N$ has a convergent subsequence. Prove it. A sequence is bounded if $|\mathbf{a}_n| \leq M$, $\forall n$, for some number M.

Solution. Take $N = 2$ for simplicity. $\{\mathbf{a}_n\}$ is bounded implies $\{x_n\}$ and $\{y_n\}$ are bounded by the previous exercise. Pick a convergent subsequence $\{x_{n_k}\}$ from $\{x_n\}$. As $\{y_{n_k}\}$ is a bounded sequence, pick a convergent sequence $\{y_{n_{k_j}}\}$ from $\{y_{n_k}\}$. Then $(x_{n_{k_j}}, y_{n_{k_j}})$ is a convergent subsequence for $\mathbf{a}_n = (x_n, y_n)$.

4. The Fibonacci sequence is defined by $f_{n+1} = f_n + f_{n-1}, f_1 = f_2 = 1$. Consider the sequence $\{a_n\}$ given by $a_n = f_n/f_{n+1}$. Establish the followings:

   (a) $1/2 \leq a_n \leq 1$.

   (b) $\{a_n\}$ is a Cauchy sequence.

   (c) Find the limit of $\{a_n\}$.

   Hint: Observe $a_{n+1} = 1/(1 + a_n)$.

Solution. See Text.